Dyadic $\mathcal{H}^1$-BMO Duality

We show that: $\boxed{|(b,\phi)| \lesssim \|b\|_{BMO_\mathcal{D}(\mathbb{R}^n)} \|\phi\|_{\mathcal{H}^1_\mathcal{D}(\mathbb{R}^n)}}$,

where $\boxed{\|\phi\|_{\mathcal{H}^1_\mathcal{D}(\mathbb{R}^n)} := \|S_\mathcal{D}\phi\|_{L^1(\mathbb{R}^n)}}$.

For every $k \in \mathbb{Z}$, define the sets:
and the collection of dyadic cubes:

$$\boxed{\begin{array}{l} U_k := \{x \in \mathbb{R}^n : S_\mathcal{D}\phi(x) > 2^k\} \\ \widetilde{U}_k := \{x \in \mathbb{R}^n : M_\mathcal{D}\mathbb{1}_{U_k}(x) > \frac{1}{2}\} \end{array}}$$

$$\boxed{\mathcal{R}_k := \{Q \in \mathcal{D} : |Q \cap U_k| > \tfrac{1}{2}|Q|\}}$$ (the cubes Q s.t. $S_\mathcal{D}\phi > 2^k$ on at least "half" of Q).

Some properties of this construction:

a. $\boxed{U_{k+1} \subset U_k \text{ and } \mathcal{R}_{k+1} \subset \mathcal{R}_k}$

$x \in U_{k+1} \Rightarrow S_\mathcal{D}\phi(x) > 2^{k+1} > 2^k \Rightarrow x \in U_k$

$Q \in \mathcal{R}_{k+1} \Rightarrow \frac{1}{2}|Q| < |Q \cap U_{k+1}| \leq |Q \cap U_k| \Rightarrow Q \in \mathcal{R}_k$.

b. $\boxed{\left(\sum_{k\in\mathbb{Z}} |U_k| 2^k\right) \simeq \|S_\mathcal{D}\phi\|_1}$ This comes directly from a measure theory fact (proof later):

$$\boxed{\|f\|_{L^1(\mu)} \simeq \sum_{k\in\mathbb{Z}} 2^k \mu\{|f| > 2^k\}}$$

Remark: Since we need an estimate involving $\|S_\mathcal{D}\phi\|_1$, we will really be after an estimate involving $\sum_{k\in\mathbb{Z}} |U_k| 2^k$.

c. $\boxed{|\widetilde{U}_k| \lesssim |U_k|}$ Recall here the weak $(1,1)$ property of the dyadic maximal function:

$$\boxed{\mu\{M^\mu_\mathcal{D} f > \lambda\} \leq \frac{1}{\lambda}\|f\|_{L^1(\mu)}}$$

So:

$$|\widetilde{U}_k| = \left|\{x : M_\mathcal{D}\mathbb{1}_{U_k}(x) > \tfrac{1}{2}\}\right| \leq 2\|\mathbb{1}_{U_k}\|_{L^1} = 2|U_k|.$$

Remark: **Every $\widetilde{U}_k$ is a bounded open set, thus it is the countable disjoint union $\widetilde{U}_k = \bigcup \widetilde{Q}$, where $\widetilde{Q}$ are the maximal dyadic cubes contained in $\widetilde{U}_k$.** (will be needed later).

The boundedness of $\widetilde{U}_k$ follows immediately from $|\widetilde{U}_k| \leq 2|U_k|$, and $|U_k| < \infty$, since we are assuming $\phi \in \mathcal{H}^1_\mathcal{D}$ (so $S_\mathcal{D}\phi$ is integrable).

To see that $\widetilde{U}_k$ is an open set, note that

$$M_\mathcal{D}\mathbb{1}_{U_k}(x) = \sup_{Q \ni x} \langle \mathbb{1}_{U_k} \rangle_Q = \sup_{Q \ni x} \frac{|Q \cap U_k|}{|Q|}$$

So if $x \in \widetilde{U}_k \Rightarrow M_\mathcal{D}\mathbb{1}_{U_k}(x) > \frac{1}{2} \Rightarrow \exists Q \in \mathcal{D}$ s.t. $x \in Q$ and $\dfrac{|Q \cap U_k|}{|Q|} > \dfrac{1}{2}$.

But then $M_\mathcal{D}\mathbb{1}_{U_k}(y) > \frac{1}{2}$ for all $y \in Q$, and $Q \subset \widetilde{U}_k$.

$\forall x \in \widetilde{U}_k, \exists Q \in \mathcal{D}$ s.t. $x \in Q$ and $Q \subset \widetilde{U}_k$.

d. $\boxed{\left(\bigcup_{Q\in\mathcal{R}_K} Q\right) \subset \widetilde{U}_K ,\ \forall K\in\mathbb{Z}}$

$x\in Q,\ Q\in\mathcal{R}_K \Rightarrow \langle \mathbb{1}_{U_K}\rangle_Q = \frac{|Q\cap U_K|}{|Q|} > \frac{1}{2} \Rightarrow M_{\mathcal{D}}\mathbb{1}_{U_K}(x) > \frac{1}{2} \Rightarrow x\in\widetilde{U}_K,\ \forall x\in Q,\ Q\in\mathcal{R}_K \Rightarrow \boxed{Q\subset\widetilde{U}_K,\ \forall Q\in\mathcal{R}_K}$

e. $\boxed{Q\notin \bigcup_{K\in\mathbb{Z}}\mathcal{R}_K \Rightarrow (\phi,h_Q)=0}$

Remark: This property simplifies the sum we are estimating:
$|(b,\phi)| \leq \sum_{Q\in\mathcal{D}} |(b,h_Q)|\cdot|(\phi,h_Q)| = \sum_{Q\in\bigcup_{K\in\mathbb{Z}}\mathcal{R}_K} |(b,h_Q)||(\phi,h_Q)|.$
(we only need to look at cubes in $\bigcup\mathcal{R}_K$).

Suppose $Q\notin \bigcup_{K\in\mathbb{Z}}\mathcal{R}_K \Rightarrow |Q\cap\{S_{\mathcal{D}}\phi>2^K\}| \leq \frac{1}{2}|Q|,\ \forall K\in\mathbb{Z}$

$\Rightarrow |Q\cap\{S_{\mathcal{D}}\phi\leq 2^K\}| \geq \frac{1}{2}|Q|,\ \forall K\in\mathbb{Z}$

$\Rightarrow |Q\cap\{S_{\mathcal{D}}\phi=0\}| = \left|\bigcap_{K=1}^{\infty}(Q\cap\{S_{\mathcal{D}}\phi\leq 1/2^K\})\right|$ (decreasing nest)

$= \lim_{K\to\infty} |Q\cap\{S_{\mathcal{D}}\phi\leq 1/2^K\}| \geq \frac{1}{2}|Q|>0$

$$\Rightarrow (\phi,h_Q)^2 = \int_{\{S_{\mathcal{D}}\phi=0\}} (\phi,h_Q)^2 \frac{\mathbb{1}_{Q\cap\{S_{\mathcal{D}}\phi=0\}}}{|Q\cap\{S_{\mathcal{D}}\phi=0\}|}\,dx \leq 2\int_{\{S_{\mathcal{D}}\phi=0\}} (\phi,h_Q)^2\frac{\mathbb{1}_Q}{|Q|}\,dx$$
$$\leq 2\int_{\{S_{\mathcal{D}}\phi=0\}} (S_{\mathcal{D}}^2\phi)\,dx = \boxed{0}$$

f. $\boxed{\bigcap_{K\in\mathbb{Z}}\mathcal{R}_K=\emptyset}$

Since $\phi\in\mathcal{H}^1_{\mathcal{D}}$, clearly $|\{S_{\mathcal{D}}\phi=\infty\}|=0$.
Now, suppose

$Q\in\bigcap_{K\in\mathbb{Z}}\mathcal{R}_K \Rightarrow |Q\cap\{S_{\mathcal{D}}\phi>2^K\}| > \frac{1}{2}|Q|,\ \forall K\in\mathbb{Z}$

$\Rightarrow 0 = |\{S_{\mathcal{D}}\phi=\infty\}| = \left|\bigcap_{K=1}^{\infty}(Q\cap\{S_{\mathcal{D}}\phi>2^K\})\right| = \lim_{K\to\infty}|Q\cap\{S_{\mathcal{D}}\phi>2^K\}| \geq \frac{1}{2}|Q|$

$\Rightarrow 0\geq |Q|$, contradiction!

Remark: Given that $\mathcal{R}_{K+1}\subset\mathcal{R}_K$, the collections $\{\mathcal{R}_K\setminus\mathcal{R}_{K+1}\}_{K\in\mathbb{Z}}$ are disjoint, in the sense that any $Q\in\mathcal{D}$ can belong to at most one collection $\mathcal{R}_K\setminus\mathcal{R}_{K+1}$
This further simplifies our sum:

$$\sum_{Q\in\bigcup_{K\in\mathbb{Z}}\mathcal{R}_K} |(b,h_Q)||(\phi,h_Q)| = \sum_{K\in\mathbb{Z}}\ \sum_{Q\in\mathcal{R}_K\setminus\mathcal{R}_{K+1}} |(b,h_Q)||(\phi,h_Q)|$$

The property f. is important here: any $Q\in\bigcap_{K\in\mathbb{Z}}\mathcal{R}_K$ would not be contained in any $\mathcal{R}_K\setminus\mathcal{R}_{K+1}$, and would have to be added separately.

Now we are ready to prove the result:

$$|(b,\phi)| \leq \sum_{Q\in\mathcal{D}} |(b,h_Q)||(\phi,h_Q)| = \sum_{k\in\mathbb{Z}} \sum_{Q\in\mathcal{R}_k\setminus\mathcal{R}_{k+1}} |(b,h_Q)||(\phi,h_Q)|$$

$$\leq \sum_{k\in\mathbb{Z}} \underbrace{\Big(\sum_{Q\in\mathcal{R}_k\setminus\mathcal{R}_{k+1}} (b,h_Q)^2\Big)^{1/2}}_{\lesssim \|b\|_{BMO}|\tilde{U}_k|^{1/2}} \underbrace{\Big(\sum_{Q\in\mathcal{R}_k\setminus\mathcal{R}_{k+1}} (\phi,h_Q)^2\Big)^{1/2}}_{\lesssim 2^k|\tilde{U}_k|^{1/2}}$$

$$\lesssim \|b\|_{BMO} \sum_{k\in\mathbb{Z}} 2^k|\tilde{U}_k|$$

$$\underset{ⓒ}{\lesssim} \|b\|_{BMO} \sum_{k\in\mathbb{Z}} 2^k|U_k|$$

$$\underset{ⓑ}{\simeq} \|b\|_{BMO}\|S_{\mathcal{D}}\phi\|_1 = \|b\|_{BMO}\|\phi\|_{\mathcal{H}^1}$$

$$\Big(\sum_{Q\in\mathcal{R}_k\setminus\mathcal{R}_{k+1}} (b,h_Q)^2\Big)^{1/2} \leq \Big(\sum_{Q\in\mathcal{R}_k} (b,h_Q)^2\Big)^{1/2}$$

$$\underset{ⓐ}{\leq} \Big(\sum_{\substack{Q\in\mathcal{D}\\ Q\subset\tilde{U}_k}} (b,h_Q)^2\Big)^{1/2}$$

$$= \Big(\sum_{\tilde{Q}} \underbrace{\sum_{Q\subset\tilde{Q}} (b,h_Q)^2}_{\lesssim|\tilde{Q}|\|b\|^2_{BMO}}\Big)^{1/2}$$ where $\tilde{Q}$ are the maximal dyadic cubes contained in $\tilde{U}_k$

$$\lesssim \|b\|_{BMO}\Big(\sum_{\tilde{Q}}|\tilde{Q}|\Big)^{1/2}$$

$$= \|b\|_{BMO}|\tilde{U}_k|^{1/2}$$

$$\sum_{Q\in\mathcal{R}_k\setminus\mathcal{R}_{k+1}} (\phi,h_Q)^2$$

$$Q\in\mathcal{R}_k\setminus\mathcal{R}_{k+1} \Rightarrow \begin{cases} Q\subset\tilde{U}_k \\ |Q\cap U_k| > \frac{|Q|}{2} \\ |Q\cap U_{k+1}| \leq \frac{|Q|}{2} \\ \text{or} \\ |Q\setminus U_{k+1}| \geq \frac{|Q|}{2} > 0 \end{cases}$$

$$= \int \sum_{Q\in\mathcal{R}_k\setminus\mathcal{R}_{k+1}} (\phi,h_Q)^2 \frac{\mathbb{1}_{Q\setminus U_{k+1}}}{|Q\setminus U_{k+1}|}\,dx$$

$$\underset{ⓐ}{=} \int_{\tilde{U}_k\setminus U_{k+1}} \sum_{Q\in\mathcal{R}_k\setminus\mathcal{R}_{k+1}} (\phi,h_Q)^2 \frac{\mathbb{1}_{Q\setminus U_{k+1}}}{|Q\setminus U_{k+1}|}\,dx$$

$$\leq 2\int_{\tilde{U}_k\setminus U_{k+1}} \sum_{Q\in\mathcal{R}_k\setminus\mathcal{R}_{k+1}} (\phi,h_Q)^2 \frac{\mathbb{1}_Q}{|Q|}\,dx$$

$$\leq 2\int_{\tilde{U}_k\setminus U_{k+1}} (S_{\mathcal{D}}\phi)^2\,dx$$

(off U_{k+1}, $S_{\mathcal{D}}\phi \leq 2^{k+1}$)

$$\leq 2(2^{k+1})^2|\tilde{U}_k\setminus U_{k+1}|$$

$$\lesssim 2^{2k}|\tilde{U}_k|$$